Group Equivariant Deep Learning

Lecture 2 - Steerable group convolutions

Lecture 2.4 - Group Theory | Induced representations and feature fields

Preliminaries (and intuition) for steerable group convolutions

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This mini-course serves as a module with the UvA Master AI course Deep Learning 2 https://uvadl2c.github.io/

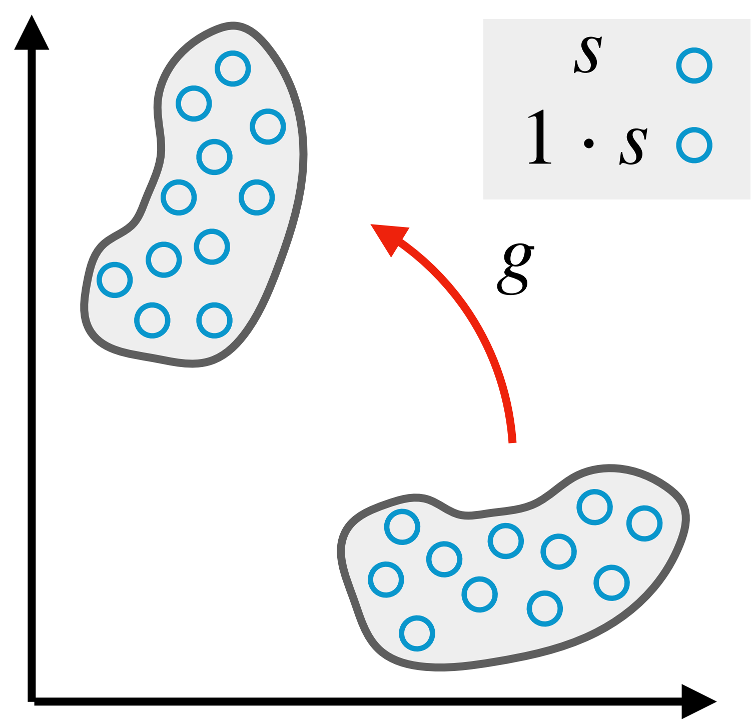
Feature field and induced representation

We call $\hat{f} : \mathbb{R}^d \to \mathbb{R}^{d_\rho}$ a feature vector field, or simply a **feature field**, if its

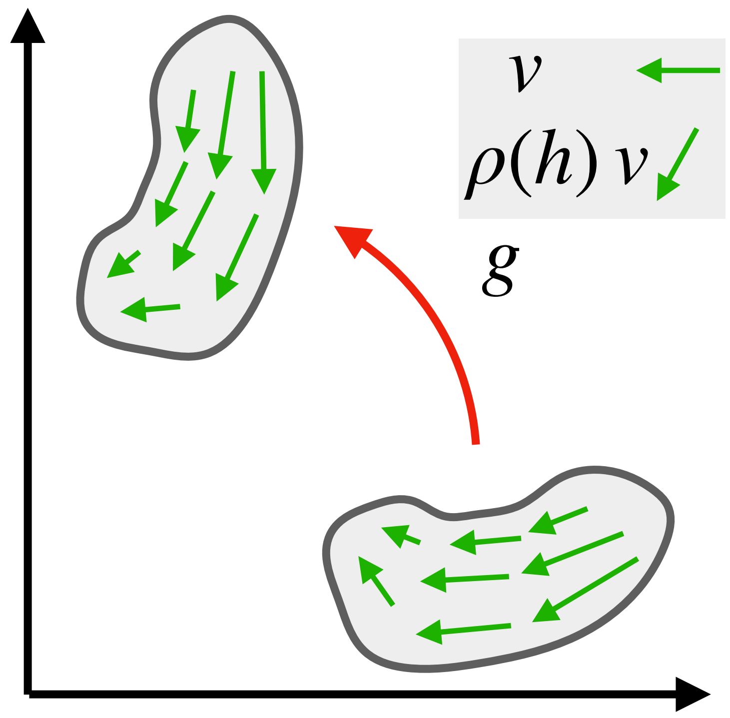
codomain	transforms via a *representation*	$\rho(h)$	of H
domain	transforms via the action	g^{-1}	of $G = (\mathbb{R}^d, +) \rtimes H$

Representation ρ defines the **type** of the field, and together with the group action of $G = (\mathbb{R}^d, +) \rtimes H$ defines the **induced representation**

$$\left(\mathrm{Ind}_H^G[\rho](\mathbf{x}, h)\,\hat{f}\right)(\mathbf{x}') := \rho(h)\,\hat{f}(h^{-1}(\mathbf{x}' - \mathbf{x}))$$



Scalar field ($\rho(h) = 1$)

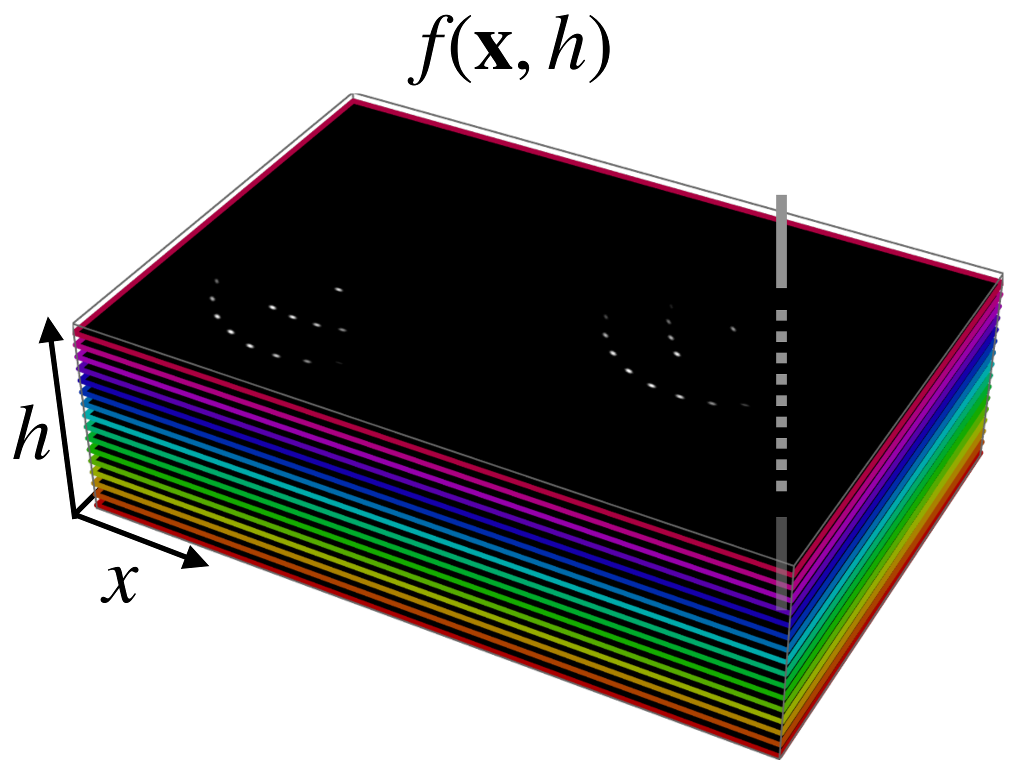


vector field

Figure adapted from: Weiler, M., & Cesa, G. (2019). General e (2)-equivariant steerable cnns. NeurIPS
See also https://github.com/QUVA-Lab/e2cnn

Feature field and induced representation

Regular G feature maps: $f(\mathbf{x}, h)$ considered so far can be considered feature fields.



$$(\mathscr{L}_g f)(\mathbf{x}', h') = f(h^{-1}(\mathbf{x}' - \mathbf{x}), h^{-1}h)$$

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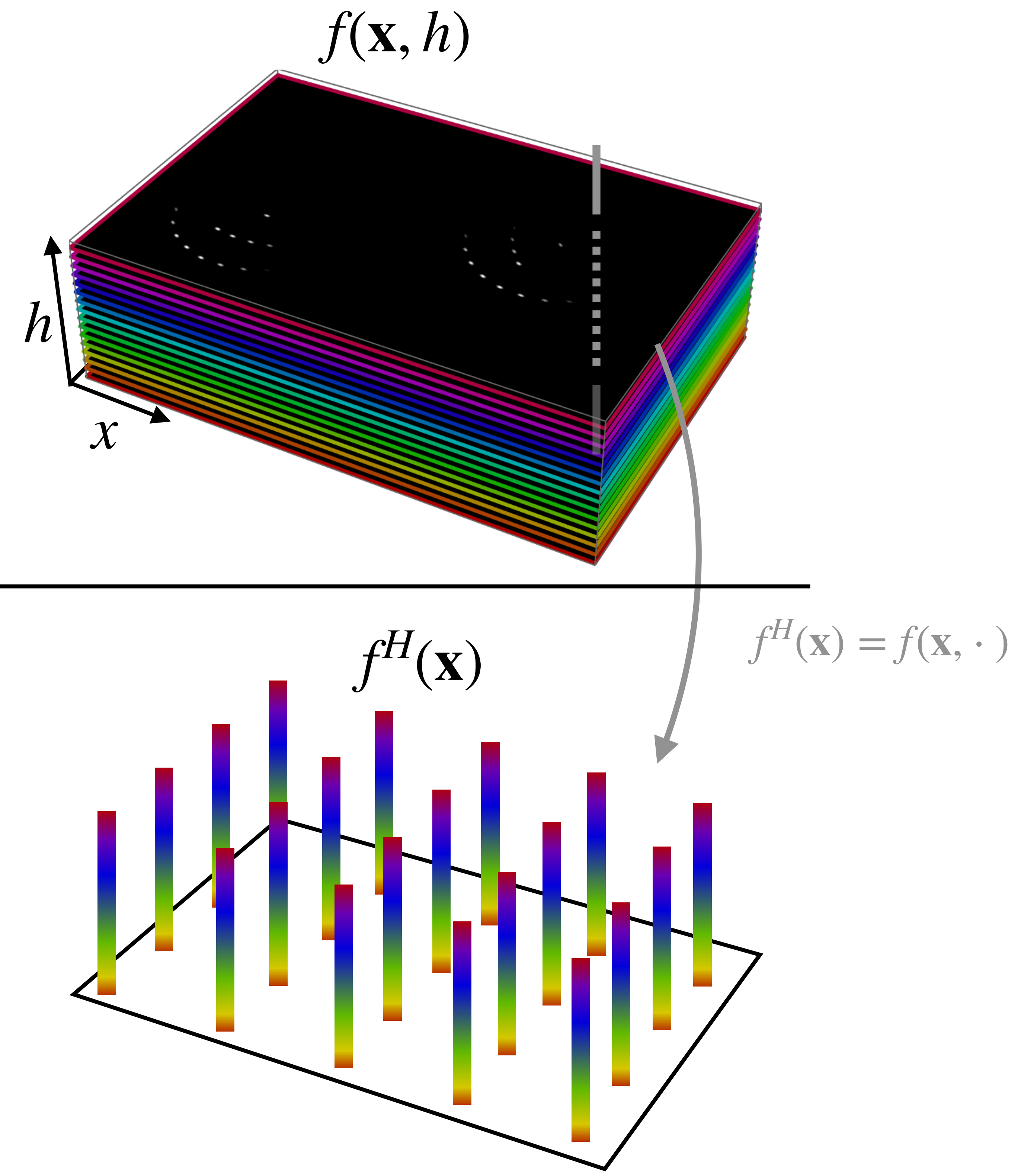

Regular H feature fields: Let $f^H(\mathbf{x}) = f(\mathbf{x}, \cdot\,)$ be the field of functions $f^H(\mathbf{x}) : H \rightarrow \mathbb{R}$ on the subgroup H, then the functions (**fibers**) transform via the regular representation $\mathscr{L}_h^H$ (recall. $\mathscr{L}_h^H f(h') = f(h^{-1}h')$)

$$(\mathscr{L}_g f)(\mathbf{x}', h') \iff \left(\mathrm{Ind}_H^G[\ \mathscr{L}_h^H\ ](\mathbf{x}, h) f^H\right)(\mathbf{x}')$$

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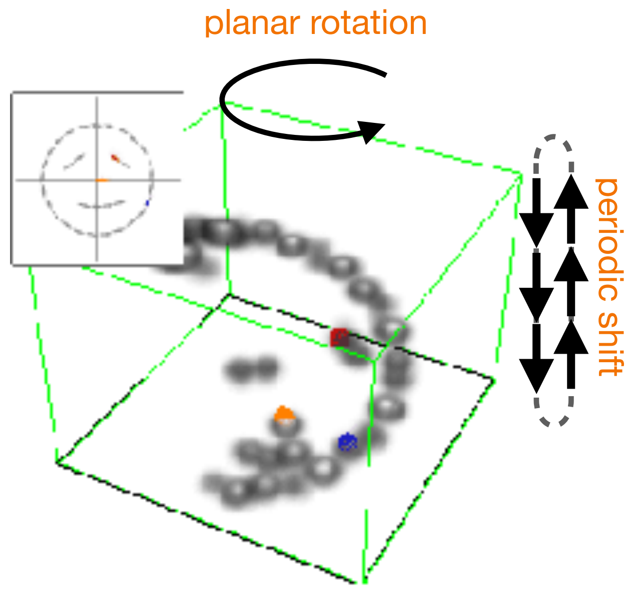

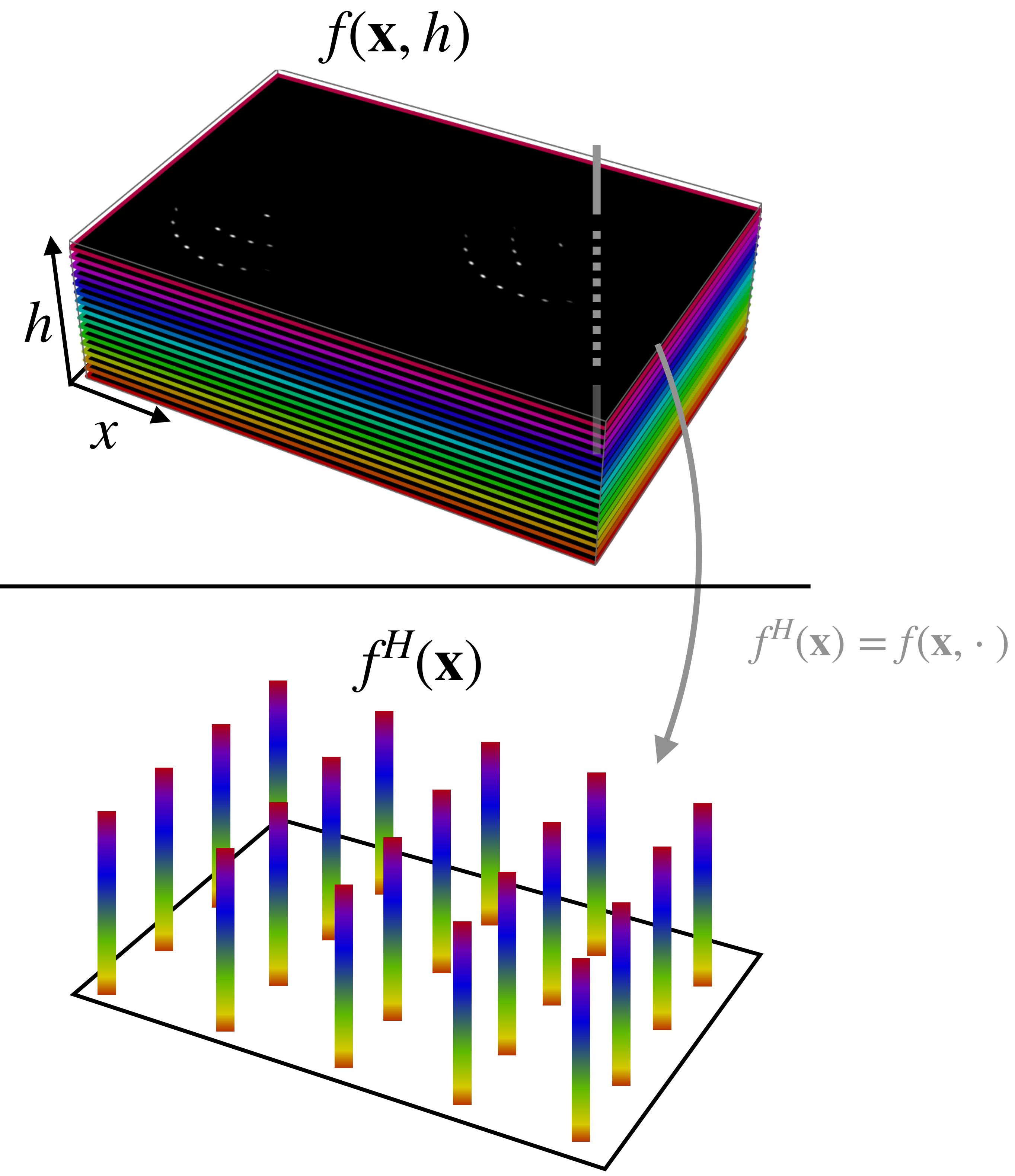

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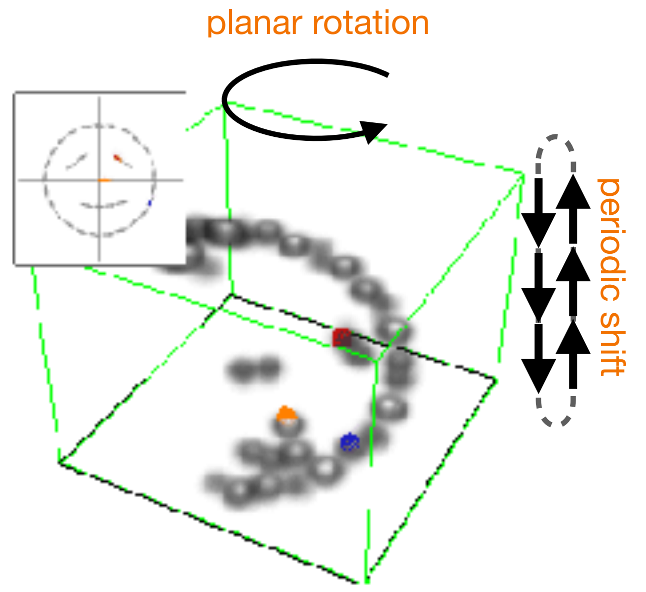

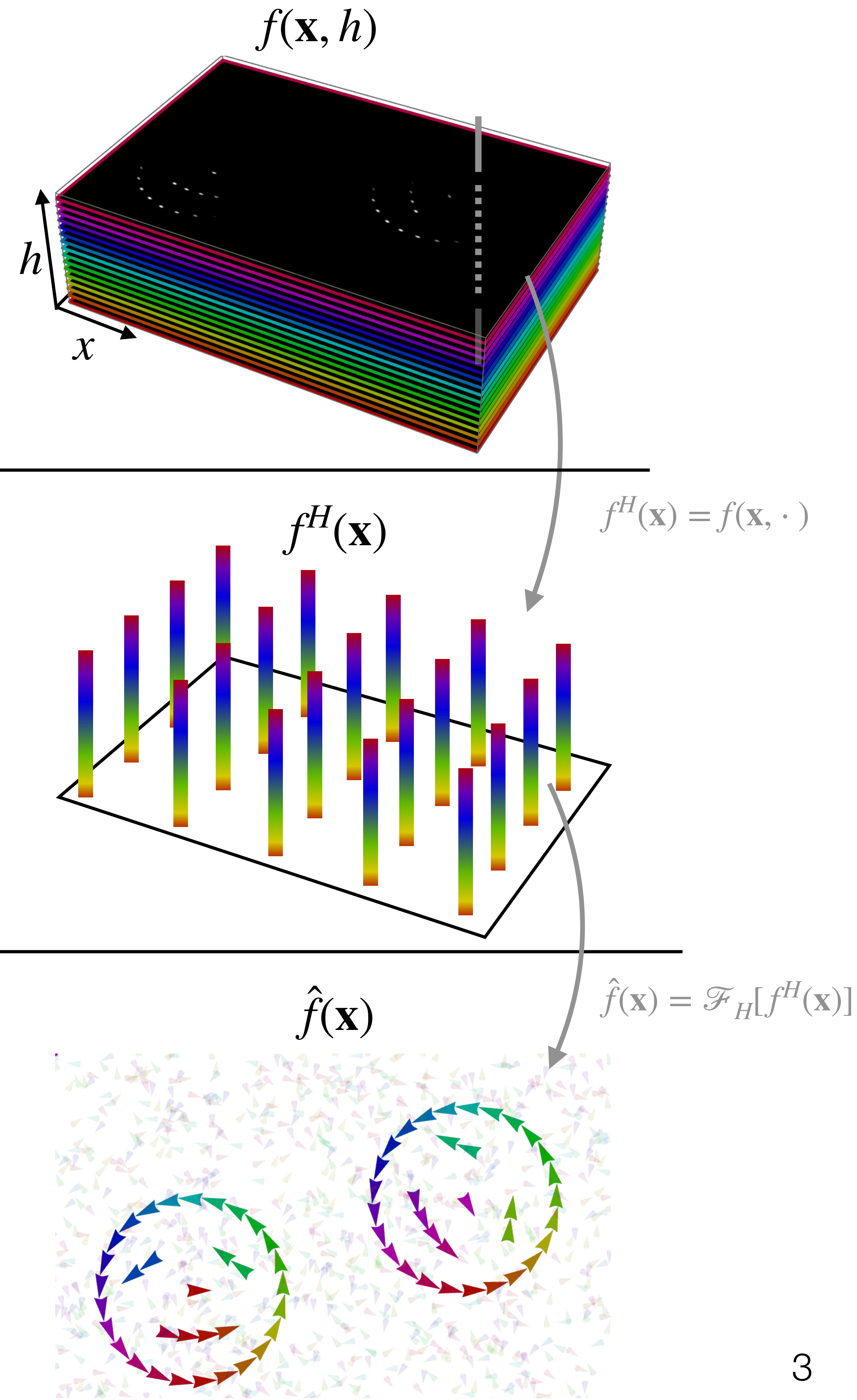

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Steerable H feature fields: Since the fibers $f^H(\mathbf{x})$ are functions on H we can represent them via their Fourier coefficients $\hat{f}(\mathbf{x}) = \mathscr{F}_H[f^H(\mathbf{x})]$. These vectors of coefficients transform via irreps $\rho(h) = \bigoplus_l \rho_l(h)$

$$(\mathscr{L}_g f)(\mathbf{x}', h') \iff \left(\mathrm{Ind}^G_H[\ \mathscr{L}^H_h\ ](\mathbf{x}, h) \hat{f}\right)(\mathbf{x}') \iff \left(\mathrm{Ind}^G_H[\ \rho(h)\ ](\mathbf{x}, h) \hat{f}\right)(\mathbf{x}')$$